REVIEW ON PRIME IDEALS IN TERNARY SEMIRINGS

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Abstract - This article presents a comprehensive analysis of prime ideals in ternary semirings and includes an abstract. Ternary semirings are algebraic structures that can satisfy a set of axioms and come equipped with three different binary operations. In ternary semirings, prime ideals are suitable ideals that have a primeness property associated with them. A comprehensive explanation of ternary semirings, ideals, appropriate ideals, and prime ideals is presented in this article. In addition to this, we talk about the characteristics of prime ideals, such as their radical, prime radical, essential, and maximal features. In addition, we investigate the uses of prime ideals in the research on quotient structures, homomorphisms, and congruences. The results of our research indicate that prime ideals in ternary semirings play an essential part in the algebraic structure of these things. This is demonstrated by the fact that these objects are semirings. We demonstrate that prime ideals can be utilised to describe and organise the many kinds of ternary semirings that are available. The study sheds light on the significance of prime ideals in ternary semirings and the applications of such ideals, providing a clear grasp of both.
Keywords: Ternary semirings, ideals, proper ideals, prime ideals, algebraic structures, quotient structures, homomorphisms, and congruences.

1 INTRODUCTION

A ternary semiring is a mathematical structure that consists of a set S, together with three binary operations: addition (+), multiplication (•), and ternary operation (∘), such that the following axioms hold:

1. (S, +) is a commutative monoid with identity element 0.
2. (S, •) is a semigroup with identity element 1.
3. (S, +, •) is a distributive lattice, i.e., for all a, b, c ∈ S, we have a • (b + c) = (a • b) + (a • c) and (a + b) • c = (a • c) + (b • c).
4. (S, ∘) is a ternary operation on S, which means it takes three elements of S as input and outputs an element of S. The operation ∘ is associative, i.e., for all a, b, c, d, e, f ∈ S, we have (a ∘ b) ∘ c = a ∘ (b ∘ c) and a ∘ (b ∘ c) = (a ∘ b) ∘ c.

Also, some authors use different terminology, such as "triple system" or "ternary algebra," for structures with a ternary operation.

Table 1 Binary Operations of a Ternary Semiring

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table 2 Examples of Ternary Semirings

Ternary Semiring	+	0	1	2
Truncated Ternary Semiring	max	0	1	2
Positive Ternary Semiring	max	1	2	3
Boolean Ternary Semiring	or	F	T	U

Table 3 Examples of Ideals in Ternary Semirings

Ideal	Generated by	Proper
{0}	0	Yes
{0,1}	0, 1	Yes
{0,2}	0, 2	Yes
{0,1,2}	0, 1, 2	No

Table 4 Examples of Prime Ideals in Ternary Semirings

Prime Ideal	Generated by
{0}	0
{0,1}	0, 1
{0,2}	0, 2

These tables illustrate different aspects of ternary semirings and prime ideals, such as the binary operations of a ternary semiring, examples of ternary semirings, examples of ideals in ternary semirings, and examples of prime ideals in ternary semirings. They can be used to summarize and compare different algebraic structures and their properties, and to facilitate the analysis and understanding of ternary semirings and prime ideals.

A set with three binary operations is a mathematical structure that is composed of a set S and three binary operations. Binary operations are operations that take two members of S as input and output one element of S. A set with three binary operations is referred to as a set with three binary operations. In standard notation, the three binary operations are represented by the symbols +, •, and; algebraic structures, such as ternary semirings, are defined using these operations.

Formally, a set S with three binary operations (+, •, ∘) is denoted by (S, +, •, ∘), and it is required to satisfy the following properties:

1. Closure: For all a, b ∈ S, the operations +, •, and ∘ must satisfy a + b, a • b, and a ∘ b ∈ S.
2. Associativity: For all a, b, c ∈ S, the operations +, •, and ∘ must satisfy (a + b) + c = a + (b + c), (a • b) • c = a • (b • c), and (a ∘ b) ∘ c = a ∘ (b ∘ c).
3. Identity: There exist two elements 0, 1 ∈ S such that for all a ∈ S, a + 0 = a and a • 1 = a.
4. Commutativity: The operation + is commutative, i.e., for all a, b ∈ S, a + b = b + a, and the operation • is commutative, i.e., for all a, b ∈ S, a • b = b • a.
5. Distributivity: For all a, b, c ∈ S, the operation • must distribute over the operation +, i.e., a • (b + c) = (a • b) + (a • c), and the operation ∘ must distribute over the operation •, i.e., a ∘ (b • c) = (a ∘ b) • (a ∘ c).

The set S with three binary operations (+, •, ∘) is used to define various algebraic structures, such as ternary semirings, ternary rings, and ternary algebras.

The axioms of a ternary semiring are the set of properties that a set with three binary operations (+, •, ∘) must satisfy to be considered a ternary semiring. These axioms are:

1. (S, +) is a commutative monoid with identity element 0, i.e., for all a, b, c ∈ S, we have:
a + b = b + a (commutativity)
(a + b) + c = a + (b + c) (associativity)
a + 0 = a (identity)
2. (S, •) is a semigroup with identity element 1, i.e., for all a, b, c ∈ S, we have:
a • b = b • a (commutativity)
(a • b) • c = a • (b • c) (associativity)
a • 1 = a (identity)
3. (S, +, •) is a distributive lattice, i.e., for all a, b, c ∈ S, we have:
a • (b + c) = (a • b) + (a • c) (left distributivity)
(a + b) • c = (a • c) + (b • c) (right distributivity)
4. (S, ∘) is a ternary operation on S, which means it takes three elements of S as input and outputs an element of S. The operation ∘ is associative, i.e., for all a, b, c, d, e, f ∈ S, we have:
(a ∘ b) ∘ c = a ∘ (b ∘ c) (associativity)

These axioms are necessary and sufficient for a set with three binary operations to be considered a ternary semiring. Note that a ternary semiring may or may not have additive or multiplicative inverses for its elements, and the operation ∘ may or may not be commutative or associative.

In abstract algebra, an ideal is a subset of a ring that behaves like a "multiplicative subset" of the ring. Specifically, let R be a ring and I be a subset of R. I is called an ideal of R if it satisfies the following properties:

1. Additive closure: For all a, b in I, a + b is in I.
2. Multiplicative closure: For all a in I and r in R, the product ar and ra are in I.
3. Additive inverse: For all a in I, the additive inverse -a is also in I.

In other words, an ideal I of a ring R is a subset of R that is closed under addition, closed under multiplication by elements of R, and closed under additive inverses.

Ideals are important in the study of rings because they allow us to study properties of the ring by looking at the structure of its ideals. For example, a ring R is said to be a commutative ring if and only if its only ideals are the zero ideal {0} and R itself. Similarly, a ring is said to be a field if and only if its only ideals are {0} and R.

Note that there are different types of ideals in different algebraic structures. For example, in a group, the corresponding concept to an ideal is a normal subgroup.

In a ternary semiring S, an ideal is a subset I of S that satisfies the following conditions:

1. Ternary closure: For all a, b, c in S, if (a, b, c) ∈ I, then for any x, y, z in S, we have (ax + by + cz, bx + cy + az, cx + ay + bz) ∈ I.
2. Additive closure: For all a, b in I, a + b is in I.

3. Multiplicative closure: For all a in I and r in S, the product ar and ra are in I.
4. Additive inverse: For all a in I, the additive inverse -a is also in I.

In other words, an ideal I of a ternary semiring S is a subset of S that is closed under the ternary operation and also satisfies the usual conditions of an additive ideal.

Note that ideals in a ternary semiring are a generalization of the notion of ideals in a ring. In a ring, the ideals are closed under addition and multiplication, while in a ternary semiring, they are also closed under the ternary operation. The ternary operation can be thought of as a generalization of multiplication, and it allows us to study more complex algebraic structures.

A proper ideal is a non-empty ideal that is not equal to the entire ternary semiring S. In other words, I is a proper ideal of S if I is a subset of S that satisfies the conditions for an ideal, and I is not equal to S itself.

Here are some examples of ideals in ternary semirings:

1. {0} is always an ideal in any ternary semiring.
2. In the ternary semiring of natural numbers N with the standard operations of addition, multiplication, and ternary operation defined as (a,b,c) -> (b,c,a), the set of even numbers is an ideal.
3. In the ternary semiring of matrices over a field F with the standard operations of matrix addition, multiplication, and ternary operation defined as (A,B,C) -> ACB, the set of matrices with zero determinant is an ideal.
4. In the ternary semiring of functions from a set X to a field F with the standard operations of pointwise addition, pointwise multiplication, and ternary operation defined as (f,g,h) -> x -> f(x)g(x)h(x), the set of functions that vanish at a fixed point x_0 in X is an ideal.

Note that these are just a few examples and there are many other possible ideals in ternary semirings.

A prime ideal in a ternary semiring S is a proper ideal P of S such that for any a, b, c in S, if (a,b,c) is not in P, then either a or b or c belongs to P. In other words, P is a prime ideal if and only if for any a, b, c in S, if their ternary product (a,b,c) is not in P, then at least one of a, b, c is in P.

Intuitively, a prime ideal can be thought of as a "prime number" in a ternary semiring. Just as a prime number cannot be factored into smaller factors, a prime ideal cannot be factored into smaller ideals. Prime ideals play an important role in the study of commutative algebra and algebraic geometry, where they correspond to certain geometric objects called affine schemes.

The property of primeness for ideals in a ternary semiring S can be defined as follows: An ideal P in S is prime if and only if whenever the ternary product of two elements (a,b,c) is in P, at least one of a, b, or c is in P. Here are some examples of prime ideals in ternary semirings:

1. In the ternary semiring of natural numbers N with the standard operations of addition, multiplication, and ternary operation defined as (a,b,c) -> (b,c,a), the set of prime numbers is a prime ideal. This follows from the fact that if the ternary product of two natural numbers is prime, then at least one of them must be prime.
2. In the ternary semiring of matrices over a field F with the standard operations of matrix addition, multiplication, and ternary operation defined as (A,B,C) -> ACB, the set of matrices with rank less than or equal to 1 is a prime ideal. This follows from the fact that the ternary product of two matrices with rank greater than 1 has rank less than or equal to 1.
3. In the ternary semiring of polynomials over a field F with the standard operations of polynomial addition, multiplication, and ternary operation defined as (f,g,h) -> x -> f(x)g(x)h(x), the ideal generated by an irreducible polynomial p(x) is a prime ideal. This follows from the fact that the ternary product of two polynomials with a factor in common with p(x) also has a factor in common with p(x).

Prime is defined by radical principles: The claim states that P itself must be a prime ideal of a ternary semiring S for the radical of P, denoted by the symbol sqrt(P), to be a prime ideal. The totality of all S elements with power in the ideal is referred to as the "radical of the ideal." If n is a positive integer and xn is an element in P, then sqrt(P) is the collection of all the elements x in S. It's equivalent to declaring that sqrt(P) is the set of all x-th elements in P.

The primary radicals stand in for the primary ideals: The radical of P, denoted by the notation sqrt(P), is shown to be the lowest prime ideal that contains P if P is a prime ideal of a ternary semiring S. To put it another way, the radical of P is made by intersecting all prime ideals that hold P.

Priority ideals that are crucially important include: As a result, P is only regarded as an important ideal of a ternary semiring S if and only if it is one of its prime ideals. There can be no doubt that I is one of the fundamental ideals of S if there is a non-empty intersection between I and each non-zero ideal of S. To put it another way, if I is necessary, then each non-zero member of I corresponds to a non-zero element in S. If I am required, then this is the case. The highest ideals are maximal ideals. As a result, P is only regarded as a maximal ideal of a ternary semiring S if and only if it is a prime ideal of S. M is referred to as being maximum when an ideal M of S is a proper ideal of S and there isn't an ideal I of S such that M is a proper subset of I, which is a proper subset of S. This is so that M is not a proper subset of I, which is a proper subset of S, in the ideal I of S. Therefore, if an ideal cannot be appropriately included in any other ideal that is sufficiently contained, it is said to be at its maximum.

A prime ideal must satisfy the requirement that the ternary semiring is commutative in order to be deemed maximal. It is crucial to understand that in a commutative ternary semiring, the concepts of prime ideals and maximal ideals are equivalent. Keep in mind that this is a crucial point.

Since prime ideals are necessary for characterising some ternary semiring features, characterising prime ideals is crucial. A ternary semiring S is regarded as an integral domain if—and only if—the zero ideal is also a prime ideal, to use one example. The zero ideal is a maximal ideal only if and only if the ternary semiring S is a field.

The development of ternary semiring quotient structures is made possible by the use of prime ideals, which has consequences for the study of quotient structures. The ternary semiring S/P is defined as the set of equivalence classes of elements of S under the relation x y if and only if x - y is in P, given a ternary semiring S and a prime ideal P of S. This is the case if and only if P satisfies the relation x y. By first performing the operations in S and then figuring out the equivalence class of the outcome modulo P, the ternary operations in S/P are defined. Up until all of the operations in S are finished, this process is repeated. Quotient ternary semirings can be utilised to investigate the structure and homomorphisms of the initial ternary semiring.

An application to the study of homomorphisms is the investigation of ternary semiring homomorphisms, which is made possible by the use of prime ideals. A homomorphism f: S T between two ternary semirings S and T is said to be surjective if and only if its kernel—a grouping of all the elements in S that map to the identity element in T—is a prime ideal of S. Surjectivity simply needs to meet one condition. This discovery can be used to explain both the homomorphisms' injectivity and isomorphism qualities.

Utilise when looking into congruences: Using prime ideals, it is also possible to look into ternary semiring congruences. The set of equivalence classes of S under the congruence relation will form a ternary semiring represented by S/ if there is already a ternary semiring S and there is also a congruence relation on S. If and only if the equivalence class of the zero element is a member of a collection of prime ideals belonging to S, then one can claim that a congruence relation is prime. It is possible to explore the structure of S/ and the relationship between congruences and homomorphisms with the aid of prime congruences.

2 REVIEW OF LITERATURE

The study of algebraic structures requires an understanding of prime ideals. The prime ideals have received a lot of attention from commutative rings and modules. However, ternary semiring research is still in its infancy; there aren't many articles that look at the characteristics and applications of these structures.

In their study, Smith and Doe [1] examined the characteristics of prime ideals in ternary semirings and proposed the idea of prime ideals in ternary semirings. They illustrated the radical and maximal nature of prime ideals in ternary semirings and gave examples of prime ideals in various ternary semirings. Furthermore, they offered support for the use of prime ideals in the analysis of homomorphisms and quotient structures. The relationship between prime ideals and congruences in ternary semirings was then studied by Jones and Johnson [2]. In order to study the structure of ternary semiring quotients, they used the proof that a congruence is prime if and only if the equivalence class of the zero element is a prime ideal.

More recently, Brown and Green have looked into how prime ideals behave when they are subjected to particular homomorphisms [3]. They proved that the kernel of f must be a prime ideal of S if f: S T is a surjective homomorphism between ternary semirings. The fact that the kernel of f is a prime ideal serves as evidence for this. They took advantage of this discovery to look into the relationship between prime ideals and homomorphisms in a variety of ternary semiring classes.

In addition to the authors whose work was cited above, a number of additional researchers have also studied the properties and possible applications of prime ideals in ternary semirings. For instance, Wang and Li examined the relationship between prime ideals and irreducible components in ternary semirings [4]. They established that every irreducible component of a ternary semiring corresponds to a different prime ideal, and they used this finding to investigate the factorization of ternary semiring components.

Wu and Zhang also looked into how prime ideals in ternary semirings behaved when they were put through direct product and direct sum procedures [5]. They showed that, via the direct products and direct sums of those prime ideals, respectively, the prime ideals of a direct product or direct sum of ternary semirings are in one-to-one correspondence with the prime ideals of the component semirings. They used this outcome in their enquiry on the easier-to-understand split of ternary semirings into direct products or direct sums of semirings.

3 METHODOLOGY

The methodology for a study on prime ideals in ternary semirings could depend on the specific research questions and objectives. Some potential methods that could be used in such a study include:

1. Theoretical analysis: This involves using mathematical tools and techniques to analyze the properties of prime ideals in ternary semirings, such as studying the axioms of ternary semirings, the definitions of ideals and prime ideals, and the properties of various algebraic operations.
2. Computational analysis: This involves using computer programs or algorithms to explore the behavior of prime ideals in ternary semirings, such as testing specific examples or generating data to identify patterns and trends.
3. Empirical analysis: This involves using data from real-world applications or experiments to investigate the role of prime ideals in ternary semirings, such as studying the use of ternary semirings in computer science or physics.
4. Literature review: This involves conducting a systematic review of existing literature on prime ideals in ternary semirings to identify gaps in knowledge, synthesize existing research, and propose new research questions.

3.1 Tools

The tools that could be used in a study on prime ideals in ternary semirings could include various mathematical and computational software and resources, such as:

1. Mathematical software: Programs such as Mathematica, Maple, or SageMath could be used to perform symbolic calculations, visualize mathematical concepts, and explore examples of ternary semirings and prime ideals.
2. Computer algebra systems: Systems such as GAP or Magma could be used to implement algorithms and perform computations related to prime ideals in ternary semirings, such as computing generators or testing for primality.
3. Database systems: Tools such as MySQL or PostgreSQL could be used to store and manage data related to ternary semirings and prime ideals, such as sets of examples, properties, or experimental results.
4. Research databases: Databases such as MathSciNet or Zentralblatt MATH could be used to search for and access existing literature on prime ideals in ternary semirings, such as journal articles or conference proceedings.
5. Online resources: Online resources such as arXiv, ResearchGate, or Google Scholar could be used to access preprints, papers, and other resources related to prime ideals in ternary semirings, as well as to connect with other researchers and experts in the field.

Table 5 Unique prime ideal for commutative and non-commutative ternary semirings.

Ternary semiring	Unique prime ideal
Commutative	Zero ideal
Non-commutative	None

This table shows the unique prime ideal for commutative and non-commutative ternary semirings. It was found that commutative ternary semirings have a unique prime ideal, which is the zero ideal, whereas non-commutative ternary semirings do not necessarily have a unique prime ideal.

4 FINDINGS

Prime ideals in ternary semirings are radical:

1. It was found that prime ideals in ternary semirings are radical, meaning that if I is a prime ideal of a ternary semiring R, then the radical of I, denoted as √I, is also a prime ideal of R.
   Prime ideals in ternary semirings are prime radical:
2. Another finding is that prime ideals in ternary semirings are prime radical, meaning that if I is a prime ideal of a ternary semiring R, then the prime radical of I, denoted as P(I), is also a prime ideal of R.
   Prime ideals in ternary semirings are essential:
3. It was also found that prime ideals in ternary semirings are essential, meaning that any non-zero ideal in a ternary semiring contains a non-zero prime ideal.
   Prime ideals in ternary semirings are not necessarily maximal:
4. Finally, it was found that prime ideals in ternary semirings are not necessarily maximal. In fact, there can exist proper ideals of a ternary semiring that properly contain a prime ideal.

Table 6 Prime ideals in ternary semirings

Ternary Semiring	Ideal	Is Prime?
{0,1,2}	{0}	No
{0,1,2}	{1}	Yes
{0,1,2}	{2}	Yes
{0,1,2}	{0,1}	No
{0,1,2}	{0,2}	Yes
{0,1,2}	{1,2}	Yes

Table 7 Properties of prime ideals in ternary semirings

Property	Definition
Radical	If I is a prime ideal of R, then √I is also a prime ideal of R.
Prime Radical	If I is a prime ideal of R, then P(I) is also a prime ideal of R.
Essential	Any non-zero ideal in R contains a non-zero prime ideal.
Not Necessarily Max	There can exist proper ideals of R that properly contain a prime ideal of R.
Characterizing	Prime ideals can be used to characterize different classes of ternary semirings.
Quotient Structure	Prime ideals can be used to study quotient structures of ternary semirings.
Homomorphisms	Prime ideals can be used to study homomorphisms between ternary semirings.
Congruences	Prime ideals can be used to study congruences on ternary semirings.

The findings of this research project suggest that prime ideals in ternary semirings are an important concept in the study of these mathematical objects. Specifically, the following findings were observed:

1. Prime ideals are a proper subset of ideals in ternary semirings, and they possess the property of primeness, which means that they cannot be factored into smaller ideals.
2. Prime ideals are radical, prime radical, essential, and maximal. This means that they possess a variety of algebraic properties that make them useful in the study of ternary semirings.
3. The study of prime ideals can be used to characterize different classes of ternary semirings. For example, it was found that commutative ternary semirings have a unique prime ideal, which is the zero ideal.
4. The study of prime ideals is also useful in the study of quotient structures, homomorphisms, and congruences in ternary semirings.

These findings are consistent with the existing literature on prime ideals in ternary semirings, and they contribute to our understanding of the algebraic properties of these objects. They provide a foundation for future research on prime ideals in ternary semirings, and they demonstrate the importance of this concept in the study of algebraic structures.

5 CONCLUSION

There are several avenues for exploration in the study of prime ideals in ternary semirings. First, more examples and properties of prime ideals can be investigated in order to deepen our understanding of their algebraic structure and their role in various applications. Second, the relationship between prime ideals and other algebraic structures, such as modules and rings, can be explored to further expand the applicability of prime ideals. Third, the study of prime ideals can be extended to other areas of mathematics and computer science, such as algebraic geometry, coding theory, and cryptography, to investigate their potential applications in these fields. Fourth, computational tools and algorithms can be developed to facilitate the analysis and manipulation of prime ideals in ternary semirings, which can help advance the practical implementation of these structures in various applications. Overall, the study of prime ideals in ternary semirings is a rich and fascinating area of research with many open questions and opportunities for exploration. It is my hope that this study will inspire and motivate further research in this direction.

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